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On the Largest Zeroes of Orthogonal Polynomials for Certain Weights

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Abstract. The asymptotic growth of the largest zero of the orthogonal polynomials for the weights $W(x) = |x|^b \exp(-k|\log|x||^c)$ is investigated.

1. Introduction. Freud [3], [4] investigated the largest zeroes of orthogonal polynomials for weights on $(-\infty, \infty)$. Nevai and Dehesa [5] studied the sums of powers of zeroes of orthogonal polynomials. Here we investigate the asymptotic growth of the largest zeroes for the weights

$$(1.1)\quad W(x) = |x|^b \exp(-k|\log|x||^c), \quad x \in (-\infty, \infty) \quad \text{where } c > 1;\ k > 0;\ b \in (-\infty, \infty)$$

and

$$(1.2)\qquad W(x) = \begin{cases} x^b \exp(-k|\log x|^c), & x \in (0, \infty), \\ 0, & x \in (-\infty, 0], \end{cases}$$

where $c > 1$; $k > 0$; $b \in (-\infty, \infty)$.

When $c = 2$ and $b = 0$ in (1.2), $W(x)$ yields the Stieltjes-Wigert polynomials (Chihara [1, 2]), and Chihara [2] has remarked that very little is known about their zeroes.

2. Notation. Given a nonnegative measurable function $W(x)$ on $(-\infty, \infty)$ for which all moments

$$\mu_n(W) = \int_{-\infty}^{\infty} x^n W(x)\, dx, \qquad n = 0, 1, 2, \ldots,$$

exist, its orthogonal polynomials are

$$p_n(W; x) = \gamma_n(W) \prod_{j=1}^{n} \left(x - x_{jn}(W)\right), \qquad n = 0, 1, 2, \ldots,$$

satisfying

$$\int_{-\infty}^{\infty} p_n(W; x) p_m(W; x) W(x)\, dx = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$$

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We let $X_n(W) = \max\{|x_{jn}(W)| : j = 1, 2, \ldots, n\}$, $n = 1, 2, \ldots$. Further, for each positive t, ζ_t denotes the smallest possible number (if it exists) such that

$$(\zeta_t)^t W(\zeta_t) = \max\{x^t W(x): x \in (0, \infty)\}, \tag{2.1}$$

and assuming that

$$\int_0^\pi |\log W(\zeta \cos\theta)|\, d\theta < \infty, \qquad \zeta \in (0, \infty), \tag{2.2}$$

we define

$$G_\zeta(W) = \exp\left\{\pi^{-1}\int_0^\pi \log W(\zeta\cos\theta)\, d\theta\right\}, \qquad \zeta \in (0, \infty).$$

3. The Largest Zero.

LEMMA 3.1. *Let $W(x)$ be given by* (1.1). *Then*

$$\lim_{n\to\infty} \mu_{2n}(W)/\left[d(2n+b+1)^{(2-c)/(2(c-1))}\exp\{f(2n+b+1)^{c/(c-1)}\}\right] = 1,$$

where $d = 2\{2\pi(c-1)^{-1}(kc)^{1/(1-c)}\}^{1/2}$ and $f = (c-1)(c^c k)^{1/(1-c)}$.

Proof.

$$\begin{aligned} \mu_{2n}(W) &= 2\int_1^\infty x^{2n+b}\exp(-k(\log x)^c)\, dx + O(n^{-1}) \\ &= 2(ck^{1/c})^{-1}\int_0^\infty \exp(-v + v^{1/c}X)v^{1/c-1}\, dv + O(n^{-1}), \end{aligned} \tag{3.1}$$

where $X = (2n+b+1)k^{-1/c}$ and $x = \exp((v/k)^{1/c})$. Now apply the asymptotic formula for the integral in (3.1), given in Olver [6, p. 84, Ex. 7.3]. □

Following is our main result.

THEOREM 3.2. *Let $W(x)$ be given by* (1.1). *Then*

(i) $$\lim_{n\to\infty}\left(\frac{kc}{2n}\right)^{1/(c-1)}\log X_n(W) = 1.$$

(ii) $$\lim_{n\to\infty}\left(\frac{kc}{2n}\right)^{1/(c-1)}\log\{\gamma_{n-1}(W)/\gamma_n(W)\} = 1.$$

Proof. (i) By Lemma 3 in Freud [3, p. 95],

$$\begin{aligned} \log X_n(W) &\geq (\log\mu_{2n-2}(W) - \log\mu_{2n-4}(W))/2 \\ &= (f/2)\{(2n+b-1)^{c/(c-1)} - (2n+b-3)^{c/(c-1)}\} + O(n^{-1}) \\ &\qquad \text{(by Lemma 3.1)} \\ &= (f/2)(2n)^{c/(c-1)}\{c(c-1)^{-1}n^{-1} + O(n^{-2})\} + O(n^{-1}) \\ &= (2n/kc)^{1/(c-1)} + O(n^{(2-c)/(c-1)}). \end{aligned} \tag{3.2}$$

Next, for any $\zeta > 0$ and $A > 1$, Theorem 2 in Freud [4, p. 52] shows that

$$X_n(W) \leq A\zeta + \frac{4}{3\pi}\left(\frac{2}{\zeta}\right)^{2n-1} G_\zeta^{-1}(W)\int_{A\zeta}^\infty x^{2n-1}W(x)\, dx. \tag{3.3}$$

Freud states this under the additional assumption that $W(x)$ is positive in $(-\infty, \infty)$, but his proof is valid if (2.2) holds. It is easily seen that for some positive constant K_0, independent of ζ,

$$G_\zeta(W) \geq K_0^{-1} W(\zeta), \qquad \zeta \in (0, \infty). \tag{3.4}$$

Then taking $\zeta = \zeta_{2(n+s)}$ and $A = 2^{n/s}$ where $s \in (0, \infty)$, we obtain, from (2.1), (3.3) and (3.4),

$$\begin{aligned} X_n(W) &\leq A\zeta_{2(n+s)} + \frac{4K_0}{3\pi}\left(\frac{2}{\zeta_{2(n+s)}}\right)^{2n-1} \zeta_{2(n+s)}^{2(n+s)} \int_{A\zeta_{2(n+s)}}^{\infty} x^{-1-2s}\,dx \\ &= \zeta_{2(n+s)}\left[2^{n/s} + K_0(3\pi s)^{-1}\right]. \end{aligned} \tag{3.5}$$

Next, for large $t \in (0, \infty)$, ζ_t is a root of $d[x^t W(x)]/dx = 0$ so $\log \zeta_t = [(t + b)/kc]^{1/(c-1)}$. Taking $s = n^\delta$ in (3.5) where

$$0 < \delta < 1 \quad \text{and} \quad 1 - \delta(c-1)^{-1}, \tag{3.6}$$

we obtain

$$\log X_n(W) \leq \left[(2n + 2n^\delta + b)/kc\right]^{1/(c-1)} + n^{1-\delta}\log 2 + o(1). \tag{3.7}$$

The result follows from (3.2), (3.6) and (3.7).

(ii) follows from (i) and Theorem 1 in Freud [3, p. 91]. □

Since

$$\{X_n(W)\}^m \leq \sum_{j=1}^{n} |x_{jn}(W)|^m \leq n\{X_n(W)\}^m, \qquad m > 0, n = 1, 2, \ldots,$$

we deduce that, for $m > 0$,

$$\lim_{n\to\infty} \left(\frac{kc}{2n}\right)^{1/(c-1)} \log\left\{\sum_{j=1}^{n} |x_{jn}(W)|^m\right\} = m,$$

which provides a contrast to the results of Nevai and Dehesa [5, Theorem 1].

COROLLARY 3.3. *Let* $W(x)$ *be given by* (1.2). *Then the conclusions* (i), (ii) *of Theorem* 3.2 *remain true.*

Proof. Let

$$W^*(x) = |x|\, W(x^2) = |x|^{2b+1} \exp(-k_1 |\log|x||^c), \qquad x \in (-\infty, \infty),$$

where $k_1 = k2^c$. Then, by Theorem 3.2,

$$\lim_{n\to\infty} \left(\frac{k_1 c}{4n}\right)^{1/(c-1)} \log X_{2n}(W^*) = 1, \tag{3.8}$$

$$\lim_{n\to\infty} \left(\frac{k_1 c}{4n}\right)^{1/(c-1)} \log\{\gamma_{2n-j-1}(W^*)/\gamma_{2n-j}(W^*)\} = 1, \qquad j = 0, 1.$$

Further, the substitution $x = u^2$ yields $p_n(W; u^2) = p_{2n}(W^*; u)$ and hence

$$X_n(W) = \{X_{2n}(W^*)\}^2; \qquad \gamma_n(W) = \gamma_{2n}(W^*), \tag{3.9}$$

and the conclusions follow from (3.8) and (3.9). □

For real b, and fixed positive k, let

$$W_b(x) = \begin{cases} k\pi^{-1/2}x^b \exp\left(-k^2(\log x)^2\right), & x \in (0,\infty), \\ 0, & x \in (-\infty, 0]. \end{cases}$$

Wigert [7] explicitly found $p_n(W_0;\, x)$, $n = 1, 2, \ldots$, while Chihara [2] constructed discrete solutions of the moment problem corresponding to W_0, which provided some information regarding the distribution of $\{x_{nj}(W_0)\}_{n,j}$. Using the relation

$$W_b(x) = \alpha^{b^2} W_0\left(x/\alpha^{2b}\right), \qquad x \in (-\infty, \infty),$$

where $\alpha = \exp(1/4k^2)$, it follows that

$$p_n(W_b;\, x) = \alpha^{-b(b+2)/2} p_n\left(W_0;\, x/\alpha^{2b}\right), \qquad n = 1, 2, \ldots,$$

and hence the results of Wigert [7] and Chihara [2] for $W_0(x)$ generalize to $W_b(x)$, any $b \in (-\infty, \infty)$.

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1. T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

2. T. S. Chihara, "A characterization and a class of distribution functions for the Stieltjes-Wigert polynomials," *Canad. Math. Bull.*, v. 13, 1970, pp. 529–532.

3. G. Freud, "On the greatest zero of an orthogonal polynomial. I," *Acta Sci. Math. (Hungar.)*, v. 34, 1973, pp. 91–97.

4. G. Freud, "On the greatest zero of an orthogonal polynomial. II," *Acta Sci. Math. (Hungar.)*, v. 36, 1974, pp, 49–54.

5. P. G. Nevai & J. S. Dehesa, "On asymptotic average properties of zeroes of orthogonal polynomials," *SIAM J. Math. Anal.*, v. 10, 1979, pp. 1184–1192.

6. F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, New York, 1974.

7. S. Wigert, "Sur les polynomes orthogonaux et l'approximation des fonctions continues," *Ark. Mat. Astronom. Fysik*, v. 17, 1923.